جامعة بنها

كلية الحاسبات والذكاء الإصطناعي

Probability and Statistics

Lecture 05

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Announcement

Quiz (1)

In Lecture 6

22/3/2023

Covers: Lec 1, 2, 3, and 4

Chapter 2: Random Variable

- Discrete Random Variables.
- Probability Mass Functions.
- Cumulative Distribution Functions.
- Discrete R.V. (Mean and Variance).
- Continuous Random Variables.
- Probability Density Functions.
- Continuous R.V. (Mean and Variance).
- Joint Probability Distributions.

Continuous R. V. (1/3)

Continuous Random Variable:

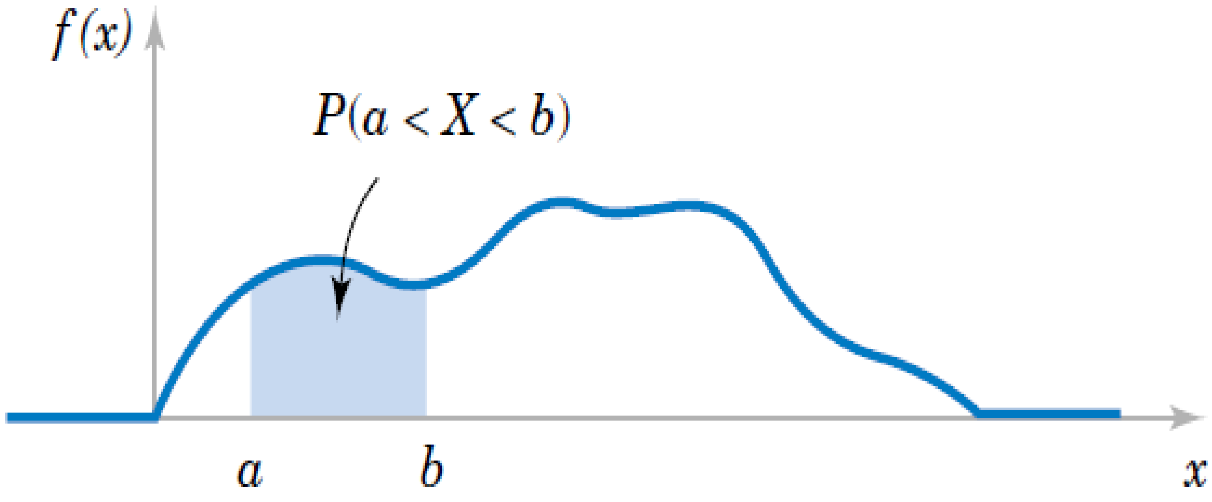
If the range space R_X of the random variable X is an interval or a collection of intervals, X is called a *continuous random variable*.

A continuous random variable has a probability of **0** of assuming *exactly* any of its values. Consequently, its probability distribution cannot be given in tabular form.

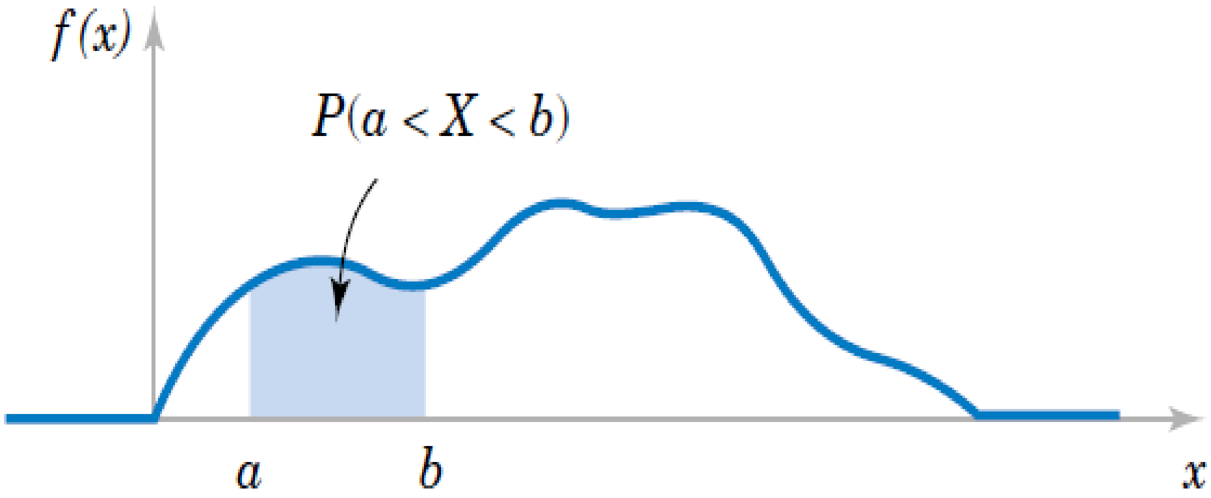
Continuous R. V. (2/3)

Example:

If we talk about the probability of selecting a person who is at least 163 centimeters but not more than 165 centimeters tall. Now we are dealing with an interval rather than a point value of our random variable.



Continuous R. V. (3/3)



If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx =$ area under $f(x)$ from a to b

for any a and b

Prob. Density Functions (2/6)

Definite Integral:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_1^3 x^2\,dx$$

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \left[\frac{3^3}{3}\right] - \left[\frac{1^3}{3}\right] = \left[9 - \left(\frac{1}{3}\right)\right] = \frac{26}{3}$$

Prob. Density Functions (3/6)

Example1:

Suppose that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{e}^{-\boldsymbol{x}}$ for $\boldsymbol{x} > 0$

Check the probability density function, then determine the following probabilities:

1. $P(X < 1)$
2. $P(1 \leq X < 2.5)$
3. $P(X = 3)$
4. $P(X \geq 3)$

Example1 – Answer (1/5)

Check the probability density function:

$$\int_{0}^{\infty} e^{-x}\, dx$$

$$\int_{0}^{\infty} e^{-x}\, dx = -e^{-x}\Big|_{0}^{\infty} = \left(-e^{-\infty}\right) - \left(-e^{0}\right) = 0 + 1 = 1$$

Prob. Density Functions (4/6)

Example1 – Answer (2/5)

1) $P(X < 1)$

$$P(X < 1) = \int_{0}^{1} e^{-x}\, dx = -e^{-x}\Big|_{0}^{1} = \left(-e^{-1}\right) - \left(-e^{0}\right)$$

$$= -0.367879 + 1 = 0.632121$$

Prob. Density Functions (4/6)

Example1 – Answer (3/5)

2) $\boldsymbol{P(1 \le X < 2.5)}$

$$\boldsymbol{P(1 \le X < 2.5) = \int_{1}^{2.5} e^{-x}\, dx = -e^{-x} \Big|_{1}^{2.5}}$$

$$\boldsymbol{= \left(-e^{-2.5}\right) - \left(-e^{-1}\right) = -0.082085 + 0.367879}$$

$$\boldsymbol{= 0.285794}$$

Prob. Density Functions (4/6)

Example1 – Answer (4/5)

3) $\boldsymbol{P(X=3)}$

$$\boldsymbol{P(X} = 3) = \boldsymbol{0}$$

Prob. Density Functions (4/6)

Example1 – Answer (5/5)

4) $P(X \geq 3)$

$$P(X \geq 3) = \int_{3}^{\infty} e^{-x}\, dx = -e^{-x}\Big|_{3}^{\infty} = \left(-e^{-\infty}\right) - \left(-e^{-3}\right)$$

$$= 0 + 0.049787 = 0.049787$$

Prob. Density Functions (5/6)

Example2:

Suppose that the error in the reaction temperature, in °C (Celsius), for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Example2 – Answer (1/2)

Check the probability density function:

$$\int_{-1}^{2} \frac{x^2}{3}\,\boldsymbol{dx}$$

$$\int_{-1}^{2} \frac{x^2}{3}\,\boldsymbol{dx} = \frac{x^3}{9}\Big|_{-1}^{2} = \left(\frac{(2)^3}{9}\right) - \left(\frac{(-1)^3}{9}\right) = \frac{8}{9} + \frac{1}{9} = \mathbf{1}$$

Prob. Density Functions (6/6)

Example2 – Answer (2/2)

2) $\boldsymbol{P(0 < X \leq 1)}$

$$\boldsymbol{P(0 < X \leq 1)} = \int_{0}^{1} \frac{x^2}{3}\, \boldsymbol{dx} = \frac{x^3}{9}\Big|_0^1$$

$$= \left(\frac{(1)^3}{9}\right) - \left(\frac{(0)^3}{9}\right) = \frac{1}{9} + \frac{0}{9} = \boldsymbol{\frac{1}{9}}$$

Cumulative Distribution Fun. (1/4)

Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$$

for $-\infty < x < \infty$.

Probability Density Function from the Cumulative Distribution Function

Given $F(x)$,

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

Cumulative Distribution Fun. (2/4)

Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$$

for $-\infty < x < \infty$.

$$P(a < X < b) = F(b) - F(a) \text{ and } f(x) = \frac{dF(x)}{dx},$$

if the derivative exists.

Cumulative Distribution Fun. (3/4)

Example1:

Find the cumulative distribution function $F(x)$ and use it to evaluate $P(0 < X \leq 1)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Cumulative Distribution Fun. (4/4)

Example1 – Answer (1/3)

Find the cumulative distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} \frac{u^2}{3} \, \boldsymbol{du}$$

$$\frac{x^2}{3}, \quad -1 < x < 2,$$

Cumulative Distribution Fun. (4/4)

Example1 – Answer (1/3)

Find the cumulative distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} \frac{u^2}{3}\, du = \int_{-1}^{x} \frac{u^2}{3}\, du$$

$$\frac{x^2}{3}, \quad -1 < x < 2,$$

Example1 – Answer (2/3)

Find the cumulative distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} \frac{u^2}{3}\, \boldsymbol{du} = \int_{-1}^{x} \frac{u^2}{3}\, \boldsymbol{du} = \frac{u^3}{9}\Big|_{-1}^{x} = \left(\frac{(x)^3}{9}\right) - \left(\frac{(-1)^3}{9}\right)$$

$$= \frac{x^3}{9} + \frac{1}{9} = \frac{x^3 + 1}{9}$$

Cumulative Distribution Fun. (4/4)

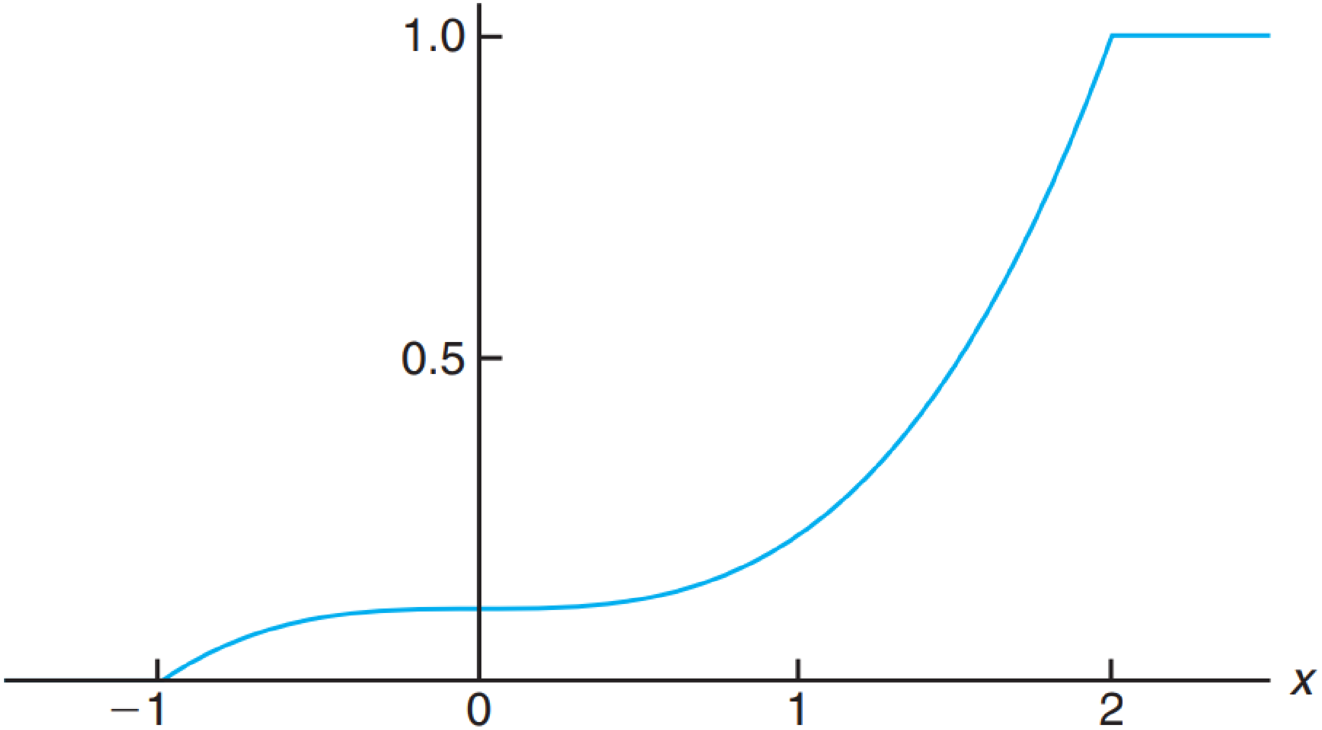
Example1 – Answer (2/3)

Find the cumulative distribution function $F(x)$

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x^3+1}{9}, & -1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

Example1 – Answer (2/3)

Find the cumulative distribution function $F(x)$



Cumulative Distribution Fun. (4/4)

Example1 – Answer (3/3)

Find the cumulative distribution function $F(x)$ and use it to evaluate $P(0 < X \leq 1)$.

$$F(x) = \frac{x^3 + 1}{9}$$

$$P(0 < X \leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Mean and Variance (1/3)

Mean and Variance

Suppose that X is a continuous random variable with probability density function $f(x)$. **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance (2/3)

Example1:

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Mean and Variance (3/3)

Example1 – Answer (1/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Mean and Variance (3/3)

Example1 – Answer (1/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-1}^{2} \frac{x^3}{3} dx = \frac{x^4}{12}\Big|_{-1}^{2}$$

Mean and Variance (3/3)

Example1 – Answer (1/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(X) = \int_{-1}^{2} \frac{x^3}{3} dx = \frac{x^4}{12}\bigg|_{-1}^{2} = \frac{(2)^4}{12} - \frac{(-1)^4}{12} = \frac{15}{12}$$

Mean and Variance (3/3)

Example1 – Answer (2/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$$E(X^2) = \int_{-1}^{2} \frac{x^4}{3} dx = \frac{x^5}{15}\Big|_{-1}^{2} = \frac{(2)^5}{15} - \frac{(-1)^5}{15} = \frac{33}{15}$$

Mean and Variance (3/3)

Example1 – Answer (3/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$E(X) = \frac{15}{12}$$

$$E(X^2) = \frac{33}{15}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{33}{15} - \left(\frac{15}{12}\right)^2 = 0.6375$$

Mean and Variance (3/3)

Example1 – Answer (3/3)

Find the expected value of X, $E(X)$ and the variance $V(X)$.

$$E(X) = \frac{15}{12}$$

$$E(X^2) = \frac{33}{15}$$

Standard Deviation (σ)
$= \sqrt{0.6375} = 0.798$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{33}{15} - \left(\frac{15}{12}\right)^2 = 0.6375$$

Joint Probability Distributions

Definition:

If X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution.

Joint Prob. Mass Fun. (1/11)

Joint Probability Mass Function:

If X and Y are two discrete random variables, the joint probability mass function is denoted as $f_{XY}(x,y)$, satisfies

1) $f_{XY}(x,y) \geq 0$

2) $\sum_X \sum_Y f_{XY}(x,y) = 1$

3) $f_{XY}(x,y) = P(X = x, Y = y)$

Joint Prob. Mass Fun. (2/11)

Example1:

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Joint Prob. Mass Fun. (2/11)

Example1 – Answer (1/2)

$$S = \{HH, HT, TH, TT\}$$

X	2	1	1	0	number of heads
Y	0	1	1	2	number of tails

Joint Prob. Mass Fun. (2/11)

Example1 – Answer (2/2)

$S = \{HH, HT, TH, TT\}$

X	2	1	1	0
Y	0	1	1	2

number of heads

number of tails

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

joint probability mass function $f_{XY}(x, y)$

Joint Prob. Mass Fun. (3/11)

Example2:

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Find:

1) $f_{XY}(1,2) = P(X = 1,\ Y = 2)$
2) $f_{XY}(2,0) = P(X = 2,\ Y = 0)$
3) $P(X = 1,\ Y \le 2)$
4) $P(Y = 2)$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (1/4)

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

1) $f_{XY}(1,2) = P(X = 1,\ Y = 2)$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (1/4)

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

1) $f_{XY}(1,2) = P(X = 1,\ Y = 2) = 0$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (2/4)

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

2) $f_{XY}(2,0) = P(X = 2,\ Y = 0)$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (2/4)

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

2) $f_{XY}(2,0) = P(X = 2,\ Y = 0) = 1/4$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (3/4)

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

3) $P(X = 1,\ Y \leq 2)$

Example2 – Answer (3/4)

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

3) $P(X = 1,\ Y \leq 2) = 0 + \frac{2}{4} + 0 = \frac{2}{4}$

Joint Prob. Mass Fun. (3/11)

Example2 – Answer (4/4)

y \ x	0	1	2
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

4) $P(Y = 2)$

Example2 – Answer (4/4)

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

4) $P(Y = 2) = \frac{1}{4} + 0 + 0 = \frac{1}{4}$

Marginal Probability Distributions

The marginal distributions of the random variable X alone is:

$$f_X(x) = \sum_y f_{XY}(x, y)$$

The marginal distributions of the random variable Y alone is:

$$f_Y(y) = \sum_x f_{XY}(x, y)$$

Example3:

A coin is tossed two times. Calculate the joint probability mass function of the discrete random variables X and Y, where X denote the number of heads appear and Y denote the number of tails appear.

Find:

1) $f_X(x)$

2) $f_Y(y)$

Joint Prob. Mass Fun. (5/11)

Example3 – Answer (1/4)

y \ x	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0

Find:

1) $f_X(x)$

Example3 – Answer (1/4)

$y \backslash x$	**0**	**1**	**2**
0	0	0	1/4
1	0	2/4	0
2	1/4	0	0
$f_X(x)$	**1/4**	**2/4**	**1/4**

Find:

1) $f_X(x)$

Joint Prob. Mass Fun. (5/11)

Example3 – Answer (2/4)

x	0	1	2
$f_X(x)$	1/4	2/4	1/4

Find:

1) $f_X(x)$

Example3 – Answer (3/4)

y \ x	**0**	**1**	**2**
0	0	0	$1/4$
1	0	$2/4$	0
2	$1/4$	0	0

Find:

2) $f_Y(y)$

Joint Prob. Mass Fun. (5/11)

Example3 – Answer (3/4)

y \ x	**0**	**1**	**2**	$f_Y(y)$
0	0	0	1/4	**1/4**
1	0	2/4	0	**2/4**
2	1/4	0	0	**1/4**

Find:

2) $f_Y(y)$

Joint Prob. Mass Fun. (5/11)

Example3 – Answer (4/4)

y	0	1	2
$f_Y(y)$	1/4	2/4	1/4

Find:

2) $f_Y(y)$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #5: https://www.youtube.com/watch?v=8X8D2ONdSK4&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=6 From 00:36:40

https://www.youtube.com/watch?v=crwvc-eW8t8&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=7

Thank You

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